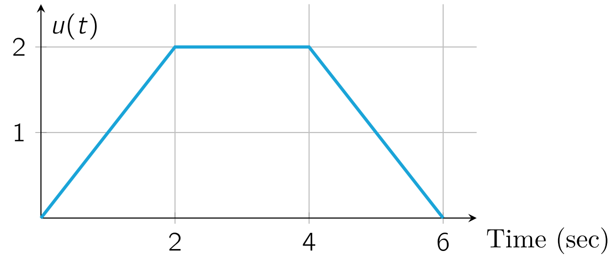
Control Systems : Set 1 : Transfer functions - Solutions

Prob 1 | Consider a system with transfer function given by

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

and the signal $u(t)$ shown in the figure below.



a) Write the Laplace transform of the signal $u(t)$.

> Define the signal $g(t) = \begin{cases} 0 & t < 0 \\ t & t > 0 \end{cases}$
>
> We can decompose the input signal $u(t)$ into the sum of four signals
>
> $$u(t) = g(t) - g(t-2) - g(t-4) + g(t-6)$$
>
> The Laplace transform is then (by the time shift propety)
>
> $$U(s) = \frac{1}{s^2}\left(1 - e^{-2s} - e^{-4s} + e^{-6s}\right)$$

b) What is the Laplace transform of the output if the signal $u(t)$ is applied to the system $G(s)$?

> In the Laplace domain, the output is the product of the input and the
>
> $$\begin{aligned} Y(s) &= G(s) \cdot U(s) \\ &= \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \cdot \frac{1}{s^2}\left(1 - e^{-2s} - e^{-4s} + e^{-6s}\right) \end{aligned}$$

Prob 2 | Find the Laplace transforms of the following functions

Note that you can use Matlab to compute Laplace transforms and check your work, for example:

```
>> syms t
>> laplace(4*sin(6*t) + cos(2*t+3))

ans =
```

24/(s^2 + 36) - (2*sin(3) - s*cos(3))/(s^2 + 4)

a) $f(t) = 1 + 5t$

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \mathcal{L}\{1\} + 5\mathcal{L}\{t\} \\ &= \frac{1}{s} + 5\frac{1}{s^2} \\ &= \frac{s+5}{s^2}\end{aligned}$$

b) $f(t) = (t+1)^2$

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \mathcal{L}\left\{t^2 + 2t + 1\right\} \\ &= \frac{2!}{s^3} + 2\frac{1}{s^2} + \frac{1}{s} \\ &= \frac{2 + 2s + s^2}{s^3}\end{aligned}$$

c) $f(t) = 4\cos 6t$

$$\begin{aligned}\mathcal{L}\{f(t)\} &= 4\mathcal{L}\{\cos 6t\} \\ &= 4\frac{s}{s^2 + 36}\end{aligned}$$

d) $f(t) = t^2 + e^{-2t}\sin 3t$

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \mathcal{L}\left\{t^2\right\} + \mathcal{L}\left\{e^{-2t}\sin 3t\right\} \\ &= \frac{2}{s^3} + \frac{3}{(s+2)^2 + 9}\end{aligned}$$

Prob 3 | Find the inverse Laplace transforms of the following functions

Note that you can use Matlab to compute inverse Laplace transforms and check your work, for example:

```
>> syms s
>> ilaplace(s/(s^2+3) + 1/s^4)

ans =

cos(3^(1/2)*t) + t^3/6
```

a) $F(s) = \dfrac{1}{s(s+1)}$

Partial fraction expansion

$$\begin{aligned} F(s) &= \frac{1}{s(s+1)} = \frac{c_1}{s} + \frac{c_2}{s+1} \\ c_1 &= 1 \\ c_2 &= -1 \\ \mathcal{L}^{-1}\{F(s)\} &= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{-1}{s+1}\right\} \\ &= 1(t) + e^{-t}1(t) \end{aligned}$$

where $1(t)$ is the unit step signal.

b) $F(s) = \dfrac{5}{s(s+1)(s+5)}$

Partial fraction expansion

$$\begin{aligned} F(s) &= \frac{5}{s(s+1)(s+5)} = \frac{c_0}{s} + \frac{c_1}{s+1} + \frac{c_2}{s+5} \\ c_0 &= \left.\frac{5}{(s+1)(s+5)}\right|_{s=0} = 1 \\ c_1 &= \left.\frac{5}{s(s+5)}\right|_{s=-1} = -\frac{5}{4} \\ c_2 &= \left.\frac{5}{s(s+1)}\right|_{s=-5} = \frac{1}{4} \\ F(s) &= \frac{1}{4\,(s+5)} - \frac{5}{4\,(s+1)} + \frac{1}{s} \end{aligned}$$

Compute the inverse transform of each term

$$f(t) = \frac{1}{4}e^{-5t} - \frac{5}{4}e^{-t} + 1$$

c) $F(s) = \dfrac{3}{s^2+4s+13}$

Approach 1: Write in a form compatible with Laplace tables

$$F(s) = \frac{3}{s^2+4s+13} = \frac{3}{(s+2)^2+3^2}$$

we can now see that the inverse transform is

$$f(t) = e^{-2t}\sin 3t$$

Approach 2: Complex partial fraction expansion

$$F(s) = \frac{3}{s^2 + 4s + 13} = -\frac{1i}{2\,(s + 2 - 3i)} + \frac{1i}{2\,(s + 2 + 3i)}$$

The pole is $p = \alpha + \beta i = -2 - 3i$, and the partial fraction coefficient is $C_1 = 1i/2$. We can now write the inverse transform as (see A.1.2 of the book)

$$\begin{aligned} f(t) &= 2|C_1| e^{\alpha t} \cos \beta t + \arg(C_1) \\ &= e^{-2t} \cos(3t - \pi/2) \end{aligned}$$

d) $F(s) = \dfrac{5}{s(s+1)(s+5)(s^2+2s+3)}$

Partial fraction expansion

$$F(s) = \frac{1}{72\,(s+5)} - \frac{5}{8\,(s+1)} + \frac{5s}{18\,(s^2+2s+3)} + \frac{1}{3s}$$

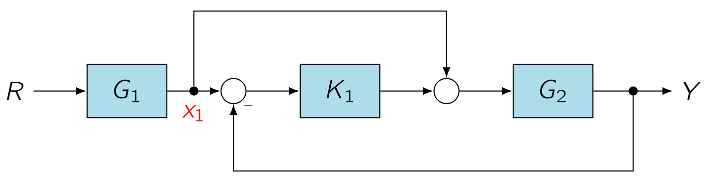
Most of the terms are simple, but consider the inverse transform of the second-order term. We re-write it to make use of the Laplace transform tables:

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{s}{s^2+2s+3}\right\} &= \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+\sqrt{2}^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{\sqrt{2}}\frac{\sqrt{2}}{(s+1)^2+\sqrt{2}^2}\right\} \\ &= e^{-t}\cos\sqrt{2}t - \frac{1}{\sqrt{2}}e^{-t}\sin\sqrt{2}t \end{aligned}$$

Take the inverse Laplace transform of each term

$$f(t) = \frac{1}{72}e^{-5t} - \frac{5}{8}e^{-t} + \frac{5}{18}e^{-t}\cos\sqrt{2}t - \frac{5}{18\sqrt{2}}e^{-t}\sin\sqrt{2}t + \frac{1}{3}$$

Prob 4 | Compute the transfer function from R to Y for the feedforward block diagram shown below.

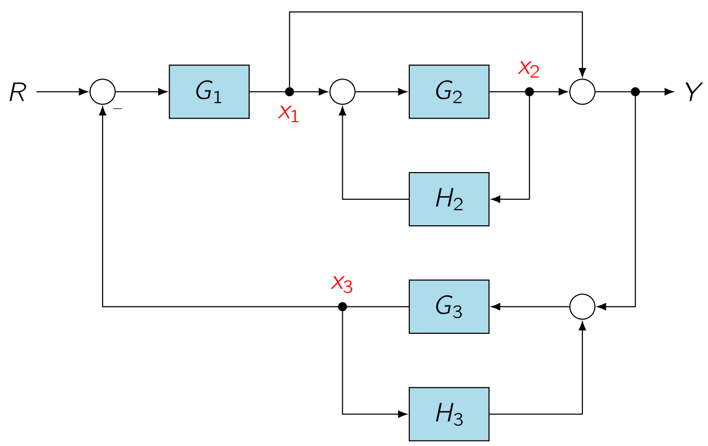


Add variables to the diagram at branch points. These are shown in red on the diagram above.
Start at the output Y and work backwards against the arrows. Open a bracket whenever you hit a sum. Repeat for each of the temporary variables defined at the branch points.

Solve the resulting equations to eliminate the temporary variables.

$$
\begin{aligned}
x_1 &= G_1 R \\
Y &= G_2(x_1 + K_1(x_1 - Y)) \\
&= G_2(G_1 R + K_1(G_1 R - Y)) \\
&= G_2 G_1 R + G_2 K_1 G_1 R - G_2 K_1 Y \\
(1 + G_2 K_1) Y &= G_1 G_2 (1 + K_1) R \\
\frac{Y}{R} &= \frac{G_1 G_2 (1 + K_1)}{1 + G_2 K_1}
\end{aligned}
$$

Prob 5 | Compute the transfer function from R to Y for the block diagram shown below.



Add variables to the diagram at branch points. These are shown in red on the diagram above.
Start at the output Y and work backwards against the arrows. Open a bracket whenever you hit a sum. Repeat for each of the temporary variables defined at the branch points.
Solve the resulting equations to eliminate the temporary variables.

$$
\begin{aligned}
x_3 &= G_3(Y + H_3 x_3) &\rightarrow\quad x_3 &= \frac{G_3}{1 - G_3 H_3} Y \\
x_1 &= G_1(R - x_3) &\rightarrow\quad x_1 &= G_1 R - \frac{G_1 G_3}{1 - G_3 H_3} Y \\
x_2 &= G_2(x_1 + H_2 x_2) &\rightarrow\quad x_2 &= \frac{G_2}{1 - G_2 H_2} x_1 = \frac{G_2 G_1}{1 - G_2 H_2} R - \frac{G_2}{1 - G_2 H_2} \frac{G_1 G_3}{1 - G_3 H_3} Y
\end{aligned}
$$

$$
\begin{aligned}
&Y = x_1 + x_2 \\
&Y = G_1 R - \frac{G_1 G_3}{1 - G_3 H_3} Y + \frac{G_2 G_1}{1 - G_2 H_2} R - \frac{G_2}{1 - G_2 H_2} \frac{G_1 G_3}{1 - G_3 H_3} Y \\
&(1 - G_3 H_3)(1 - G_2 H_2) Y \\
&\quad = G_1 (1 - G_3 H_3)(1 - G_2 H_2) R - G_1 G_3 (1 - G_2 H_2) Y + G_2 G_1 (1 - G_3 H_3) R - G_1 G_2 G_3 Y \\
&((1 - G_3 H_3)(1 - G_2 H_2) + G_1 G_2 G_3 + G_1 G_3 (1 - G_2 H_2)) Y \\
&\qquad\qquad\qquad\qquad = (G_1 (1 - G_3 H_3)(1 - G_2 H_2) + G_2 G_1 (1 - G_3 H_3)) R \\
&\frac{Y}{R} = \frac{G_1 (1 - G_3 H_3)(1 - G_2 H_2) + G_2 G_1 (1 - G_3 H_3)}{(1 - G_3 H_3)(1 - G_2 H_2) + G_1 G_2 G_3 + G_1 G_3 (1 - G_2 H_2)}
\end{aligned}
$$